COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 13 (4) (1993) 435–440

# FRACTIONALLY COLOURING TOTAL GRAPHS

K. KILAKOS and B. REED

*Received October 29, 1990*

Behzad and Vizing have conjectured that given any simple graph of maximum degree $\Delta$, one can colour its edges and vertices with $\Delta+2$ colours so that no two adjacent vertices, or two incident edges, or an edge and either of its ends receive the same colour. We show that for any simple graph $G$, $V(G)\cup E(G)$ can be fractionally coloured with $\Delta+2$ colours.

## 1. Preliminaries

The *total graph* $T(G)$ of a graph $G=(V,E)$ has vertex set $V\cup E$; two vertices of $T(G)$ are adjacent precisely if they correspond to adjacent vertices of $G$, incident edges of $G$ or an edge of $G$ and one of its ends. A long standing conjecture of Behzad [1] and Vizing [7] is that for any simple graph $G$, $T(G)$ can be coloured with at most $\Delta+2$ colours, where $\Delta$ is the maximum vertex degree of $G$. This is equivalent to saying that the vertices of $T(G)$ can be partitioned in at most $\Delta+2$ sets of mutually disjoint vertices called *stable sets.* The best known bound on the chromatic number of $T(G)$, obtained by Hind [4], is $\chi'(G)+2\lceil\chi^{1/2}(G)\rceil$, where $\chi'(G)$ is the chromatic index of $G$ and $\chi(G)$ its chromatic number. Only very special cases of the conjecture have been verified (see [6]). In this paper we formulate the general total fractional colouring problem for simple graphs and present a solution to it requiring at most $\Delta+2$ colours. In other words, we show that when fractions of stable sets are permitted, the Behzad–Vizing conjecture holds true.

Each stable set $T$ of $T(G)$ corresponds to the union of a matching $M$ and a stable set $S$ of $G$ such that $\delta(S)\cap M=\emptyset$, where $\delta(S)$ denotes the edges of $G$ with exactly one endpoint in $S$. We call such sets *total stable sets.* The family of total stable sets of $G$ is denoted by $\mathcal{T}$. A *fractional total colouring* of a simple graph $G$ is a feasible solution of the linear program

$$(1)\qquad \min\left\{w\bar{1}\ :\ w\geq 0;\ \sum_{T\ni u} w_T\geq 1,\ u\in E(G)\cup V(G),\ T\in\mathcal{T}\right\},$$

AMS subject classification codes (1991): 05 C 15, 05 C 70, 90 C 10 where $w$ is a row vector indexed by the total stable sets of $G$ and $\bar{1}=(1,\ldots,1)^T$. (Strictly speaking, fractional total colourings are feasible solutions to

$$\min\left\{w\bar{1} \;:\; w\geq 0;\; \sum_{T\ni u} w_T = 1,\; u\in E(G)\cup V(G),\; T\in\mathcal{T}\right\}.$$

However, these two linear programs are equivalent in the sense that given any feasible solution $w^1$ of the former with $w^1\bar{1}=w^*$, we can easily obtain from it a feasible solution $w^2$ if the latter with $w^2\bar{1}\leq w^*$, and vice versa.)

Ryan [5] in a recent paper gave an integer programming formulation of the total colouring problem and proved that its linear programming relaxation has its value bounded by $\Delta+2$ for any simple graph $G$. The relaxation she presented is weaker than (1) as it allows a covering of the elements of $G$ by fractions of not only total stable sets but other subgraphs of $G$ as well. We shall prove in section 2 that for any instance of a simple graph $G$, the minimum in (1) is bounded above by $\Delta+2$. In the remainder of this section we review some results, which we will need in our proof, concerning fractional edge colourings of graphs.

For any graph $G$, consider the linear program

$$\min\left\{y\bar{1} \;:\; y\geq 0;\; \sum_{M\ni e} y_M \geq c_e,\; e\in E(G),\; M\in\mathcal{M}\right\}, \tag{2}$$

where $c_e\in\mathbb{Q}_+$ and $\mathcal{M}$ is the family of matchings of $G$. A feasible solution $y$ to (2) is called a *weighted fractional colouring* of $G$. If $c=\bar{1}^T$, then $y$ is a *fractional edge colouring* of $G$. The dual of (2) is

$$\max\{cx \;:\; x\in Q_M\},$$

where

$$Q_M=\left\{x \;:\; x\in\mathbb{R}_+^{|E|};\; \sum_{e\in M} x_e\leq 1,\; M\in\mathcal{M}\right\}.$$

The theory of antiblocking polyhedra developed by Fulkerson [3], tells us that the antiblocker $A(Q_M)$ of $Q_M$ is are convex hull of the incidence vectors of matchings of J. Edmonds [2] has shown that

$$A(Q_M)=\left\{x \;:\; x\in\mathbb{R}_+^{|E|};\; \sum_{e\in\delta(v)} x_e\leq 1,\; v\in V(G);\right.$$
$$\left.\sum_{e\in E(U)} x_e\leq \lfloor |U|/2\rfloor,\; U\subset V(G),\; |U|=1\right\},$$

where $E(U)=\{e\in E(G) \;:\; e=uv,\; \{u,v\}\subseteq U\}$. Hence,

$$\max\{cx \;:\; x\in Q_M\}=\max_{\substack{v\in V(G)\\ U\subseteq V(G)}}\left\{\sum_{e\in\delta(v)} c_e,\; \Big(\sum_{e\in E(U)} c_e\Big)/\lfloor |U|/2\rfloor\right\}. \tag{3}$$

The last equality suggests the following definition. An induced subgraph $H$ of $G$ is called *overfull* if $|E(H)| > \Delta(|V(H)|-1)/2$. Note that if $H$ is overfull, then $|\delta(H)| \leq \Delta-1$ and when $G$ is simple $|V(H)| \geq \Delta+1$. Also, it is easy to check that for $c=\bar{1}^T$, the maximum in (3) is $\Delta$, unless $G$ contains an overfull subgraph in which case it can be more than $\Delta$. As a consequence, if $G$ has no overfull subgraphs, then it has a fractional edge colouring $y$ with $y\bar{1} = \Delta$. An overfull subgraph of $G$ is *minimal* if no proper subset of its vertices induces an overfull subgraph of $G$. Overfull subgraphs, particularly minimal ones, turn out to be pivotal in our treatment of the fractional total colouring problem.

## 2. A $\Delta+2$ fractional total colouring

**Theorem.** *Any simple graph $G$ can be fractionally total coloured with at most $\Delta+2$ colours.*

**Proof.** The main idea of the proof is easy to explain. A $\Delta+2$ vertex colouring of $G$ is first selected. Then, for each colour class $S_i$, $1 \leq i \leq \Delta+2$, a weighted fractional edge colouring of $G-S_i$ is obtained by solving (2). Finally, these two colourings of vertices and edges of $G$ are suitably combined to obtain the desired feasible solution to (1).

Clearly, if $G$ has maximum degree $\Delta$ then it has a $\Delta+2$ vertex colouring (just colour $V(G)$ greedily). In fact, if we can choose a $\Delta+2$ vertex colouring of $G$ such that for each colour class $S_i$, $G-S_i$ contains no overfull subgraph, then it is easy to fractionally colour $T(G)$. For each $i$, we will obtain a $\Delta$ fractional edge colouring of $G-S_i$ and then combine the matchings in this colouring with $S_i$ to obtain the total stable sets containing $S_i$. The details are explained below.

Consider a $\Delta+2$ vertex colouring of $G$ and suppose that for each colour class $S_i$, $1 \leq i \leq \Delta+2$, $G-S_i$ contains no overfull subgraph of $G$ and thus has a fractional edge colouring $y^i$ satisfying $y^i\bar{1} = \Delta$. Let $\mathcal{M}_i$ denote the family of matchings of $G-S_i$. We can now obtain a feasible solution to (1). For each $i$, $1 \leq i \leq \Delta+2$, let $T_{ij} = S_i \cup M_{ij}$ and $w_{T_{ij}} = y^i_{M_{ij}}/\Delta$ for all $M_{ij} \in \mathcal{M}_i$. For all other total stable sets $T$ of $G$, let $w_T = 0$. Clearly, each $T_{ij}$ is a total stable set of $G$. Also, for each node $v \in S_i$ we have that

$$\sum_{T \ni v} w_T = \sum_{M_{ij} \in \mathcal{M}_i} \left(y^i_{M_{ij}}/\Delta\right) = \Delta/\Delta = 1,$$

and for each edge $e \in E(G)$, with one end in $S_k$ and one end in $S_m$,

$$\sum_{T \ni v} w_T = \sum_{\substack{i=1 \\ i \notin \{k,m\}}}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} \left(y^i_{M_{ij}}/\Delta\right) \geq \sum_{\substack{i=1 \\ i \notin \{k,m\}}}^{\Delta+2} (1/\Delta) = \Delta/\Delta = 1.$$

Moreover,

$$w\bar{1} = \sum_{i=1}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} \left(y^i_{M_{ij}}/\Delta\right) = \sum_{i=1}^{\Delta+2} 1 = \Delta + 2$$

as required.

A parenthetical note. Using the same ideas, one can easily obtain a $\Delta+3$ total colouring of any simple graph $G$. Simply start with a $\Delta+3$ vertex colouring and modify the weights of the total stable sets so that now, $w_{T_{i,j}} = y^i_{M_{i,j}}/(\Delta+1)$.

It remains to prove the theorem for the case in which $G$ has no $\Delta+2$ vertex colouring such that $G-S_i$, $1 \le i \le \Delta+2$, contains no overfull subgraphs of $G$. This is equivalent to saying that there is no partitioning of $V$ into colour classes $S_1,\dots,S_{\Delta+2}$ such that $S_i \cap V(H) \neq \emptyset$ for each colour class $S_i$ and each overfull subgraphs $H$ of $G$. It turns out that we cannot colour $V$ as desired if and only if $G$ contains at least one overfull subgraph of size $\Delta+1$. In this case we will have to deal with these subgraphs separately. To begin, we show that any two such subgraphs are disjoint.

We will need the following notation. For any two disjoint subgraphs $G_1$, $G_2$ of $G$, $E(G_1,G_2)$ will denote the set of edges of $G$ with one end in $G_1$ and the other in $G_2$.

**Lemma 1.** *Let $H_1$, $H_2$ be overfull subgraphs of a graph $G$. If $|E(H_1 \cap H_2, H_1 \setminus H_2)| \le |E(H_1 \cap H_2, H_2 \setminus H_1)|$, then $H_1 \setminus H_2$ is overfull.*

**Proof.** We have that

$$\begin{gathered}|E(H_1 \cap H_2)| + |E(H_1 \cap H_2, H_1 \setminus H_2)| \\ \le (|V(H_1 \cap H_2)|\Delta - |E(H_1 \cap H_2, G \setminus (H_1 \cap H_2))|)/2 + E(H_1 \cap H_2, H_1 \setminus H_2)| \\ \le |V(H_1 \cap H_2)|\Delta/2.\end{gathered}$$

Thus,

$$\begin{gathered}|E(H_1 \setminus H_2)| = |E(H_1)| - |E(H_1 \cap H_2, H_1 \setminus H_2)| - |E(H_1 \cap H_2)| \\ > (|V(H_1)|-1)\Delta/2 - |V(H_1 \cap H_2)|\Delta/2 = (|V(H_1 \setminus H_2)|-1)\Delta/2.\end{gathered}$$

Therefore, $H_1 \setminus H_2$ is overfull. ∎

**Corollary.** *Any two minimal overfull subgraphs of a graph $G$ are disjoint.*

Corollary 2 can be used to show that if all overfull subgraphs of $G$ contain at least $\Delta+2$ vertices, then there exists a $\Delta+2$ vertex colouring of $G$ such that each overfull subgraph of $G$ contain a vertex of each colour. To this end, consider a graph $G$ with no overfull subgraphs of size $\Delta+1$ and let $H$ be a minimal overfull subgraph of $G$ (if $G$ has no such subgraph, any $\Delta+2$ colouring of $V$ suffices). If $H$ is the only minimal overfull subgraph of $G$, then we can choose a set $D$ of $\Delta+2$ vertices in $H$, give them all distinct colours and then extend this greedily to a colouring of $V$. If there is another minimal overfull subgraph in $G$, then, by induction, we can colour $V(G \setminus H)$ such that with overfull subgraph of $G \setminus H$ contains a vertex of each colour. By Corollary 2, to obtain our desired colouring, we need only extend this to a colouring of $G$ in which every colour appears in $H$. Again, we will choose a subset $D = \{v_1,\dots,v_{\Delta+2}\}$ of $H$, give these vertices distinct colours, and then extend greedily on $V(H)-D$ to complete our colouring of $V$. We choose $D$ so that it contains all the vertices of $H$ which are the ends of edges of $\delta(H)$ and such that, if $v_i$ is the endpoint of some edge of $\delta(H)$ and $i > j$ then $v_j$ is the endpoint of some edge of $\delta(H)$ (this is possible since $|\delta(H)| \le \Delta-1$).

Having coloured $\{v_1,\ldots,v_{i-1}\}$, we choose a colour for $v_i$ different from those given to $\{v_1,\ldots,v_{i-1}\}$ and any endpoint of an edge $v_iw$ with $w \in V(G\setminus H)$ as follows. Clearly, if there are no such edges we can find a colour for $v_i$. If there is such an edge then by our choice of $D$, $|\{w : w\in V(G\setminus H),\ v_iw\in E\}|\leq|\delta(G)|-(i-1)\leq\Delta-i$ and hence, there are at least three choices for the colour of $v_i$. It follows that we can obtain our desired colouring of $G$.

The above remarks allow us to restrict our attention to graphs which have at least one overfull subgraph of size $\Delta+1$. Hence, $G=G'\bigcup_{j=1}^{m} H_j$, where the $H_j$'s are disjoint overfull subgraphs of $G$ each of order $\Delta+1$ and $G'$ contains no such subgraph. Furthermore, the method of the previous paragraph, with the obvious modifications, can be used to obtain a partitioning of the vertices of $G$ into colour classes $S_1,\ldots,S_{\Delta+2}$ such that for every overfull subgraph $G$ of $G$, $S_i\cap V(H)\neq\emptyset$, $1\leq i\leq\Delta+1$, and $S_{\Delta+2}\cap V(H)\neq\emptyset$ when $|V(H)|\geq\Delta+2$.

We now describe how to obtain a fractional total colouring $w$ of $G$. Since $\delta(H_j)\leq\Delta-1$ and $|V(H_j)|=\Delta+1$, $1\leq j\leq m$, $H_j$ contains a vertex $v_j$ with all its neighbours in $H_j$. Select one such vertex from each $H_j$ and let $A$ be the set of the chosen vertices. Denote by $A_i$ the set of vertices of $A\cap S_i$, $1\leq i\leq\Delta+1$. For each colour class $S_i$, $1\leq i\leq\Delta+1$, we obtain an optimal fractional edge colouring $y^i$ of $G-S_i$, unless $A_i\neq\emptyset$. If $A_i\neq\emptyset$, then we set $c_e=1/2$ if $e\in\delta(A_i)$ and $c_e=1$ otherwise and obtain, by solving (2), an optimal weighted edge colouring $y^i$ of $G-(S_i-A_i)$ with these edge weights. Finally for $i=\Delta+2$, we define a vector $c$ on the edges of $G-S_{\Delta+2}$, with $c_e=1/2$ if $e\in\delta(A)$, $c_e=1$ otherwise, and again obtain a minimum weighted fractional edge colouring $y^{\Delta+2}$ of $G-S_{\Delta+2}$ relative to $c$.

By increasing some components of each $y^i$, if necessary, we shall assume that $y^i\bar{1}=\Delta$. We can also ensure that for any $i$, $1\leq i\leq\Delta+2$, whenever $c_e=1/2$, we have that $\sum_{M\ni e} y^i_M=c_e$ by dropping $e$ from some total stable sets. Let $\mathcal{M}_i$ denote the family of matchings of $G-(S_i-A_i)$. We are now ready to describe a fractional total colouring of $G$. For each $i$, $1\leq i\leq\Delta+2$, let

$$T_{ij}=\begin{cases} M_{ij}\cup(S_i-\{v\ :\ v\in A_i,\ \delta(v)\cap M_{ij}\neq\emptyset\}) & \text{if } 1\leq i\leq\Delta+1\\ M_{ij}\cup S_i\cup\{v\ :\ v\in A,\ \delta(v)\cap M_{ij}=\emptyset\}) & \text{otherwise}\end{cases}$$

and

$$w_{T_{ij}}=y^iM_{ij}/\Delta \quad 1\leq i\leq\Delta+2,$$

for all $M_{ij}\in\mathcal{M}_i$. For any other total stable set $T$ of $G$, $w_T=0$.

A simple inspection ensures that each $T_{ij}$ is a total stable set of $G$. Moreover, $w$ is a fractional total colouring of $G$. Indeed, consider any $v\in S_i$ and suppose that $v\in A_i$, then

$$\begin{aligned}\sum_{T\ni v} w_T &= \sum_{\substack{M_{ij}\in\mathcal{M}_i:\\ \delta(v)\cap M_{ij}=\emptyset}}\left(y^i_{M_{ij}}/\Delta\right)+\sum_{\substack{M_{\Delta+2j}\in\mathcal{M}_{\Delta+2}:\\ \delta(v)\cap M_{\Delta+2j}=\emptyset}}\left(y^{\Delta+2}_{M_{\Delta+2j}}/\Delta\right)\\ &=\Delta/(2\Delta)+\Delta/(2\Delta)=1,\end{aligned}$$

if $v \notin A_i$, then

$$\sum_{T \ni v} w_T = \sum_{M_{ij} \in \mathcal{M}_i} \left( y^i_{M_{ij}} / \Delta \right) = \Delta/\Delta = 1,$$

Also, for every $e \in \delta(A)$, with one end in $A \cap S_k$ and one end in $S_m$,

$$\sum_{T \ni e} w_T = \sum_{\substack{i=1 \\ i \neq m}}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} \left( y^i_{M_{ij}} / \Delta \right) \geq (\Delta - 1)/\Delta + 1/(2\Delta) + 1/(2\Delta) = 1$$

and for any other $e \in E(G)$, with one end in $S_k$ and the other in $S_m$,

$$\sum_{T \ni e} w_T = \sum_{\substack{i=1 \\ i \notin \{k,m\}}}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} \left( y^i_{M_{ij}} / \Delta \right) \geq \sum_{\substack{i=1 \\ i \notin \{k,m\}}}^{\Delta+2} (1/\Delta) = \Delta/\Delta = 1.$$

Finally,

$$w\bar{1} = \sum_{i=1}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} \left( y^i_{M_{ij}} / \Delta \right) = \sum_{i=1}^{\Delta+2} 1 = \Delta + 2.$$

and the proof of the theorem is complete.

We note that there are graphs for which optimal fractional total colourings contain exactly $\Delta + 2$ colours. Complete graphs with an even number of vertices are such examples.

## References

[1] M. Behzad: Graphs and their chromatic numbers, *Doctoral Thesis* (Michigan State University), (1965).

[2] J. Edmonds: Maximum matching and a polyhedron with 0, 1 vertices, *Journal of Research of the National Bureau of Standards (B)* **69** (1965), 125–130.

[3] D. R. Fulkerson: Anti-blocking polyhedra, *J. Combinatorial Theory B* **12** (1972), 50–71.

[4] H. R. Hind: An upper bound on the total chromatic number, *Graphs and Combinatorics* **6** (1990), 153–158.

[5] J. Ryan: Fractional total colouring, *Discrete Appl. Math.* **27** (1990), 287–292.

[6] N. Vijayaditya: On the total chromatic number of a graph, *J. London Math. Soc. (2)* **3** (1971), 405–408.

[7] V. G. Vizing: Some unsolved problems in graph theory (in Russian), *Uspekhi Math. Nauk.* **23** (1968), 117–134.

K. Kilakos

*Dept de Mathématique et d'Informatique*
*Université du Quebéc à Montréal,*
*Canada*
kilakos@mipsmath.math.uqam.ca

B. Reed

*Department of Mathematics*
*Carnegie Mellon University,*
*USA*
bareed@andrew.cmu.edu